On the Rank of Operators in Reflexive Algebras

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ABSTRACT

An element S of an algebra is called single if $ASB = 0$ implies $AS = 0$ or $SB = 0$. For many algebras of operators, S is single if and only if S has rank one. Here we study properties of single operators not of rank one, and conditions on the algebra which will ensure that all its single operators are of rank one.

1. INTRODUCTION

In the study of representations of a C^*-algebra, or more generally of a semisimple Banach algebra $\mathscr{A}$, elements of $\mathscr{A}$ with the property of being single (an element S of an algebra $\mathscr{A}$ is called single if the condition $ASB = 0$ for A, B in $\mathscr{A}$ implies AS or $SB = 0$) play a role imitating operators of rank one in operator algebras. For instance, it is proved in [4] that there exists a faithful representation of any C^*-algebra into the (bounded) operators on a Hilbert space in such a way that single elements, and only those, are carried to operators of rank one. For general semisimple Banach algebras, it is proved in [6] that they can be faithfully represented in the algebra of operators on a Banach space in such a way that single compactly acting elements, and only those, are carried to operators of rank one. Also it is proved in [6] that these elements completely characterize the socle of the algebra in the sense that every element of the socle is a finite sum of such elements. For the use of single elements in the study of annihilator Banach algebras see [8].

655 Avenue of the Americas, New York, NY 10010 0024-3795/90/$3.50

In yet another aspect, namely in the study of algebraic homomorphisms between reflexive algebras of operators on a normed space, single elements have also proved a useful tool. This is mainly because single elements are carried to single elements under algebraic isomorphisms. So if, in a particular operator algebra, it is known that each single element is of rank one (the converse is always true: rank one operators are single in any algebra that contains them), then the study of algebraic isomorphisms is considerably simplified. Reflexive algebras that have been looked at from this point of view include nest algebras on Hilbert spaces [16, 17], algebras of operators leaving invariant the elements of a complete atomic Boolean lattice of subspaces of a normed space [12], and more recently commutative subspace lattices [14]. On each of the first two of the abovementioned operator algebras it is proved [5, 12, 16] that single elements are of rank one. Using this, one shows that each algebraic isomorphism between a pair of such algebras is automatically continuous and spatial in the sense that it is of the form $\Phi(A)=T^{-1}AT$ for suitable T [12, 14, 16]. Note that the above cases include the algebra $\mathscr{B}(X)$ of all operators on the underlying space, and so generalize a classical result of Eidelheit in [3].

Both nest algebras and algebras of operators leaving invariant complete atomic Boolean subspace lattices are special cases of algebras leaving invariant complete and completely distributive subspace lattices (called strongly reflexive in [13]). These algebras were proved to be reflexive by Longstaff [13], thus generalizing results of Ringrose [16] and Halmos [9].

In this paper we elaborate on the study of single elements of an algebra of operators leaving invariant a strongly reflexive subspace lattice and so generalize some of the results in [5, 12, 14, 16]. Although single elements of such algebras have a behavior resembling rank one operators, and in many cases coincide with them, Moore constructed [7] (see also [14]) a single element of rank two. In the first part of the paper we show that a single element not in the radical of such an algebra is always of rank one. We also observe that, for a single element S, the topological condiiton $S \notin \operatorname{rad} \mathscr{A}$ is equivalent to the algebraic condition $S\mathscr{A}S \neq \{0\}$.

The second part of the paper gives a partial answer to Question 2 of Moore [14], which asks for conditions on the underlying lattice that ensure that every nonzero single operator is of rank one. In a case where this fails, namely in the algebra of operators leaving invariant the ordinal sum $\mathscr{L}_1 + \mathscr{L}_2$ of two complete atomic Boolean subspace lattices $\mathscr{L}_1$ and $\mathscr{L}_2$, we produce a sharp upper bound for the rank of every single element in terms of the number of atoms of $\mathscr{L}_1$ and $\mathscr{L}_2$. In particular, by a construction motivated from [14], we show the existence of single elements of any rank, including infinity. Finally we discuss the behavior of single elements in the algebra of operators leaving invariant various other lattices in which the radical of the algebra is nontrivial, but all nonzero single elements are of rank one.

2. PRELIMINARIES AND NOTATION

In what follows we denote by X a fixed (real or complex) normed space. By *subspace* we mean a *closed* subspace, and by *operator* we mean *bounded* operator. The algebra of all operators on X is denoted by $\mathscr{B}(X)$, and the set of continuous linear functionals on X by X^*. A subspace L of X is *invariant* under an operator A on X if $A(L) \subseteq L$. The set of all operators leaving invariant the elements of a collection $\mathscr{L}$ of subspaces of X is denoted by $\operatorname{Alg} \mathscr{L}$. Dually, if $\mathscr{A}$ is a subset of $\mathscr{B}(X)$, we denote by $\operatorname{Lat} \mathscr{A}$ the set of all subspaces of X that are invariant under each operator in $\mathscr{A}$. It is easy to see that for any $\mathscr{L}$ as above, the set $\operatorname{Alg} \mathscr{L}$ is a subalgebra of $\mathscr{B}(X)$ containing the identity, and if X is a Banach space, the algebra $\operatorname{Alg} \mathscr{L}$ is a Banach algebra. Dually, for any $\mathscr{A} \subseteq \mathscr{B}(X)$ the set $\operatorname{Lat} \mathscr{A}$ is a complete lattice containing X and the zero subspace (0). We refer to Radjavi and Rosenthal [15] for standard definitions and properties of invariant subspace theory.

A complete lattice of subspaces of X containing (0) and X itself is called a *subspace lattice*. In a subspace lattice the operations $L \vee M$ and $L \wedge M$ for $L, M \in \mathscr{L}$ are the *closed linear span* and the *set theoretic intersection* of L and M respectively. More generally, if $L_i \in \mathscr{L}$ $(i \in I)$, then $\bigvee_I L_i$ denotes the closed linear span of the L_i $(i \in I)$, and $\bigwedge_I L_i$ denotes their set theoretic intersection. A totally ordered subspace lattice is called a *nest*. A *complemented* lattice is a lattice with a largest element I and a smallest element 0 such that for each $L \in \mathscr{L}$ there is an $L' \in \mathscr{L}$, a *lattice complement* of L, such that $L \vee L' = I$ and $L \wedge L' = 0$. In a subspace lattice the largest and smallest elements are X and (0) respectively, and so lattice complementation amounts to subspace quasicomplementation. A lattice $\mathscr{L}$ is called *distributive* if the identity

$$L \wedge (M \vee N) = (L \wedge M) \vee (L \wedge N) \qquad (L, M, N \in \mathscr{L})$$

and its dual hold. A complemented and distributive lattice is called a *Boolean lattice*. In such a lattice complements are unique. An element L of a lattice $\mathscr{L}$ with 0 is called an *atom* if the condition $0 \subseteq K \subseteq L$ $(K \in \mathscr{L})$ implies either $K = 0$ or $K = L$. A lattice is called *atomic* if each element of the lattice is the span of the atoms it contains. Both complete nests and complete atomic Boolean subspace lattices are particular instances of subspace lattices in which the infinite distributive identity

$$\bigwedge_{a \in A} \bigvee_{b \in B} L_{a,b} = \bigvee_{f \in B^A} \bigwedge_{a \in A} L_{af(a)}$$

and its dual hold, where B^A denotes the set of all functions $f: A \to B$ (see [13]). Such complete lattices are called *completely distributive*, and completely distributive subspace lattices are called *strongly reflexive* (see [13]). Totally ordered subspace lattices are called *nests*, and the corresponding algebras of operators leaving them invariant are called *nest algebras*. We refer to Birkhoff [1] for standard definitions and properties of lattices.

If L is a subspace of a normed space X, we denote by $L^{\perp}$ the *annihilator* of L, that is, the subspace of X^* given by $L^{\perp} = \{y^* \in X^* \mid y^*(x) = 0 \ \forall x \in L\}$. Dually, if $M \subseteq X^*$ then $M_{\perp}$ is $\{x \in X \mid m^*(x) = 0 \ \forall m^* \in M\}$. Note that for subspaces we have $L = (L^{\perp})_{\perp}$. The adjoint of an operator $A \in \mathscr{B}(X)$ is denoted by A^*, and the restriction of A to a subspace L of X by $A|L$. If A belongs to a subalgebra $\mathscr{A}$ of $\mathscr{B}(X)$, then the *left annihilator* of A in $\mathscr{A}$, in symbols $\operatorname{lan}_{\mathscr{A}}(A)$ or simply $\operatorname{lan}(A)$, is the set $\{B \in \mathscr{A} \mid BA = 0\}$. Similarly for the *right annihilator* $\operatorname{ran}(A)$. The *rank* of an operator is the *algebraic* dimension of its range. If $e^* \in X^*$ and $f \in X$, then the rank one operator $x \to e^*(x)f$ on X is denoted by $e^* \otimes f$. If $\mathscr{L}$ is a subspace lattice and $L \in \mathscr{L}$, we denote by L_- the subspace $\bigvee\{K \in \mathscr{L} \mid L \not\subseteq K\}$. For example (see [13, p. 495]), if $\mathscr{L}$ is a complete atomic Boolean subspace lattice, then $L_- \neq X$ if L is an atom, in which case $L_- = L'$, the Boolean complement of L in $\mathscr{L}$. In all other cases $L_- = X$ (except $0_- = 0$). In the case of a nest, L_- is the immediate predecessor of L if there is one, or L itself if there is no such immediate predecessor.

The following two results are essentially from [13]. The first is a normed space version of a result given in [13] for Hilbert spaces (see remarks in [12]), and the second is a selection of various remarks and statements of ([13, p. 495].

LEMMA 2.1 (Longstaff [13]). *If $\mathscr{L}$ is a subspace lattice, then $e^* \otimes f$ belongs to* Alg $\mathscr{L}$ *if and only if there is a subspace $L \in \mathscr{L}$ with $L_- \neq X$ such that $f \in L$ and $e^* \in L_-^{\perp}$.*

LEMMA 2.2 (Longstaff [13]). *If $\mathscr{L}$ is a strongly reflexive subspace lattice, then for every $L \in \mathscr{L}$ we have*

(i) $L = \bigcap\{K_- \mid K \in \mathscr{L}, K \not\subseteq L\}$,
(ii) $L = \bigvee\{K \in \mathscr{L} \mid L \not\subseteq K_-\}$.

If in Lemma 2.2(ii) we take X for L, we conclude that $X = \bigvee\{K \in \mathscr{L} \mid K_- \neq X\}$, in other words, the linear manifold $X_0 = \operatorname{linear span}\{K \in \mathscr{L} \mid K_- \neq X\}$ is (norm) dense in X. Also recall that if $(L_a)_{a \in A}$ are subspaces of X, then $(\bigcap_a L_a)^{\perp}$ equals the closure in the weak* topology of X^* of the linear span of the $\{L_a^{\perp} \mid a \in A\}$. Now Lemma 2.2(i) implies $(0) = \bigcap\{K_- \mid K \in$

$\mathscr{L}$, $K \neq (0)\}$, so taking annihilators, $X^* = [\bigcap\{K_- \mid K \in \mathscr{L}, K \neq (0)\}]^\perp$. By the above remarks the set $X_1 = \text{linear span}\{K_-^\perp \mid K \in \mathscr{L}, K \neq (0)\}$ is weak* dense in X^*. It is trivial but useful to observe that in this linear span we can omit the K_- with $K_-^\perp = (0)$, so in fact $X_1 = \text{linear span}\{K_-^\perp \mid K \in \mathscr{L}, K \neq (0)$, and $K_- \neq X\}$. Using this, we can prove the following elementary lemma, previously proved for the special case of nest algebras (on a Hilbert space) [5] and for algebras of operators leaving invariant a complete atomic Boolean subspace lattice [12]. This lemma will be applied several times.

LEMMA 2.3. *Let $\mathscr{L}$ be a strong reflexive subspace lattice on a normed space, and let $A \in \operatorname{Alg}\mathscr{L}$.*

(i) *If $RA = 0$ for all rank one operators $R \in \operatorname{Alg}\mathscr{L}$, then $A = 0$.*
(ii) *If $AR = 0$ for all rank one operators $R \in \operatorname{Alg}\mathscr{L}$, then $A = 0$.*

Proof. (i): If $R \in \operatorname{Alg}\mathscr{L}$ is of rank one, then $R = e^* \otimes f$, where $e^* \in K_-^\perp$ for some $K \in \mathscr{L}$. The condition $0 = (e^* \otimes f)A = A^*e^* \otimes f$ for all rank ones of $\operatorname{Alg}\mathscr{L}$ implies $\ker A^* \supseteq K_-^\perp$, so taking the linear span over all such K, we have $\ker A^* \supseteq X_1$, where X_1 is as defined just before the statement of the lemma. The continuity of A^* with respect to the weak* topology on X^* and the weak* density of X_1 implies that A^*, and hence A, is zero.

To prove (ii) observe that the condition $0 = A(e^* \otimes f) = e^* \otimes Af$ implies $\ker A \supseteq K$ for each K with $K_- \neq X$. So $\ker A \supseteq X_0$, which is (norm) dense in X, concluding the proof. ∎

Another elementary, and no doubt well-known, lemma, which we state only because we shall refer to it many times, is the following:

LEMMA 2.4. *Let X be a normed space, let X_0 be a (norm) dense linear manifold of X, let X_1 be a weak* dense linear manifold of X^*, let $A \in \mathscr{B}(X)$, let $x, y \in X$, and let $e^*, f^* \in X^*$. Then:*

(i) *If $A|X_0$ is of finite rank, then $A(X_0) = A(X)$.*
(ii) *If $A^*|X_1$ is of finite rank, then $A^*(X_1) = A^*(X^*)$.*
(iii) *If $X_0 \cap \ker e^* = X_0 \cap \ker f^*$, then e^* and f^* are linearly dependent.*
(iv) *If $X_1 \cap \{x\}^\perp = X_1 \cap \{y\}^\perp$, then x and y are linearly dependent.*

Proof (sketch). (i): As X_0 is a linear manifold, so is $A(X_0)$. By assumption $A(X_0)$ is finite dimensional, and hence closed. So $A(X_0) = \overline{A(X_0)} \supseteq A(\overline{X_0}) = A(X)$, as required.

(ii): If $^{-*}$ denotes closure in the weak* topology, it is easy to see that for any linear manifold X_1 we have $\overline{A^*(X_1)}^* \supseteq A^*(\overline{X_1}^*)$. This last equals $A^*(X^*)$.

On the other hand, $\overline{A^*(X_1)}^*$ equals $A^*(X_1)$, since, as above, it is closed. The result now follows.

(iii): Restricting e^* and f^* to X_0, the hypothesis says that they have the same null space. By a well-known result in linear algebra these restrictions are linearly dependent; hence by continuity so are e^* and f^*.

(iv): If $\hat{}$ denotes the canonical embedding $\hat{}: X \to X^{**}$, we have $\ker \hat{x} = \{x\}^{\perp}$, so the assumption amounts to saying $X_1 \cap \ker \hat{x} = X_1 \cap \ker \hat{y}$. As in (iii), $\hat{x}|X_1$ and $\hat{y}|X_1$ are linearly dependent. But $\hat{x}, \hat{y}$ are continuous with respect to the weak* topology on X^*, and X_1 is dense in this topology, so $\hat{x}, \hat{y}$ are linearly dependent. Hence so are x and y.

3. ALGEBRAIC CONDITIONS

In this section we study single elements of Alg $\mathscr{L}$, where $\mathscr{L}$ is a strongly reflexive lattice. We show that under various assumptions on a single element we can ensure that it will be of rank one. Examples show that without these conditions, single elements may fail to be of rank one. We start with a series of lemmas that partially fulfill the desired conclusions. First however we show that the condition of being single, in strongly reflexive subspace lattices, need be verified only for part of Alg $\mathscr{L}$.

LEMMA 3.1. *Let $\mathscr{L}$ be a strongly reflexive subspace lattice and S an element of* Alg $\mathscr{L}$. *Then S is single if and only if for rank one operators R_1, R_2 of* Alg $\mathscr{L}$ *the condition $R_1SR_2 = 0$ implies R_1S or SR_2 is zero.*

Proof. If S is single, then the above condition is only a special case of the definition. Suppose then $ASB = 0$ for $A, B \in \text{Alg}\,\mathscr{L}$. If $SB \neq 0$, then by Lemma 2.3 there exists a rank one R_2 of Alg $\mathscr{L}$ such that $SBR_2 \neq 0$. For any rank one R_1 of Alg $\mathscr{L}$ we have $R_1ASBR_2 = 0$, and clearly R_1A and BR_2 are of rank one or zero. In either case the condition in the lemma implies R_1AS or SBR_2 is zero. But as $SBR_2 \neq 0$, we have for all rank ones R_1 of Alg $\mathscr{L}$ that R_1AS is zero. Applying Lemma 2.3 once again, it follows that $AS = 0$, as required. ∎

Next we state an unpublished lemma by J. A. Erdos, who kindly permitted us to include it here. The proof given is a slight modification of the original one, but is more suitable for a variant of this lemma discussed below. Also note that this lemma includes Lemma 3.2 of [14].

LEMMA 3.2 (Erdos). *Let $\mathscr{L}$ be a strongly reflexive subspace lattice and S a nonzero single element of* Alg $\mathscr{L}$. *Then there exists an M in $\mathscr{L}$ with $M_{-} \neq X$ such that $S|M$ is nonzero. Moreover, for any $L \in \mathscr{L}$ with $L_{-} \neq X$ and $S|L$ nonzero, the operator $S|L$ is of rank one.*

Proof. By Lemma 2.3 there is a rank one $R \in \text{Alg}\,\mathscr{L}$ such that $SR \neq 0$. By Lemma 2.1, R is of the form $e^* \otimes f$ where $f \in M$, $M \in \mathscr{L}$, and $M_{-} \neq X$. But then $0 \neq S(e^* \otimes f) = e^* \otimes Sf$ shows that $Sf \neq 0$, and the first part of the lemma is proved.

By Lemma 2.3 there is a rank one $T \in \text{Alg}\,\mathscr{L}$ such that $TS \neq 0$. Let now $L \in \mathscr{L}$ satisfy the condition in the statement of the lemma. We are to prove that if $x, y \in L$ then Sx and Sy are linearly dependent, so there is no loss in assuming that Sx, Sy are nonzero.

The operator TS is of rank one, so there exist scalars λ, μ, not both zero, such that $TS(\lambda x + \mu y) = \lambda(TSx) + \mu(TSy) = 0$. For any nonzero $x^* \in (L_{-})^{\perp}$ we have $x^* \otimes (\lambda x + \mu y) \in \text{Alg}\,\mathscr{L}$ and

$$TS[x^* \otimes (\lambda x + \mu y)] = x^* \otimes TS(\lambda x + \mu y) = 0.$$

However, S is single and $TS \neq 0$, so

$$x^* \otimes S(\lambda x + \mu y) = S[x^* \otimes (\lambda x + \mu y)] = 0,$$

which in turn implies $\lambda(Sx) + \mu(Sy) = 0$, and so Sx, Sy are linearly dependent. This completes the proof. ■

Adapting the ideas in the above proof, we give a kind of a dual that will be useful in Section 4. Note that if a strongly reflexive lattice $\mathscr{L}$ is a lattice of subspaces of a reflexive Banach space, then the set $\mathscr{L}^{\perp} = \{L^{\perp} \mid L \in \mathscr{L}\}$ is easily seen to be a strongly reflexive subspace lattice. In nonreflexive spaces, however, the set $\mathscr{L}^{\perp}$ of subspaces of X^* fails, usually, to be a lattice, let alone strongly reflexive. But even in the case of reflexive (or even Hilbert) spaces the elements $(L_{-})^{\perp}$ and $(L^{\perp})_{-}$ of $\mathscr{L}^{\perp}$ bear no relation: Examples show that either of the two can be strictly larger, and other examples show that they can be incomparable. So the following lemma does not seem to follow from the previous one.

LEMMA 3.3. *Let $\mathscr{L}$ be a strongly reflexive subspace lattice and S a nonzero single element of* Alg $\mathscr{L}$. *Then there exists an M in $\mathscr{L}$ with $M_{-} \neq X$ such that $S^*|M_{-}^{\perp}$ is nonzero. Moreover, for any $L \in \mathscr{L}$, with $L_{-} \neq X$ and $S^*|L_{-}^{\perp}$ nonzero, the operator $S^*|L_{-}^{\perp}$ is of rank one.*

Proof. The first part of the proof is immediate using the weak* density of the linear span of the $M_-^\perp$ with $M_- \neq X$ and the continuity of S^* with respect to the weak* topology on X^*.

Let then $L \in \mathscr{L}$ satisfy the conditions in the lemma. By Lemma 2.3 there is a $T \in \operatorname{Alg} \mathscr{L}$ of rank one such that ST is nonzero, and so of rank one. But then T^*S^* is also of rank one, so for f^*, g^* in $L_-^\perp$ there exist scalars λ, μ not both zero with $T^*S^*(\lambda f^* + \mu g^*) = \lambda(T^*S^*f^*) + \mu(T^*S^*g^*) = 0$. For a nonzero $x \in L$ we have, by Lemma 2.1, that $(\lambda f^* + \mu g^*) \otimes x \in \operatorname{Alg} \mathscr{L}$. As the adjoint of this operator is $\hat{x} \otimes (\lambda f^* + \mu g^*)$, where $\hat{x}$ is the canonical image of x in X^{**}, we have

$$T^*S^*[(\lambda f^* + \mu g^*) \otimes x]^* = T^*S^*[\hat{x} \otimes (\lambda f^* + \mu g^*)] = 0,$$

and so $[(\lambda f^* + \mu g^*) \otimes x]ST = 0$. But S is single and $ST \neq 0$, so

$$S^*(\lambda f^* + \mu g^*) \otimes x = ((\lambda f^* + \mu g^*) \otimes x) S = 0,$$

implying that $\lambda(S^*f^*) + \mu(S^*g^*) = 0$. Hence S^*f^*, S^*g^* are linearly dependent, concluding the proof that $S^*|L_-^\perp$ is of rank one. ∎

In Lemma 3.2 we showed the existence of an M in $\mathscr{L}$ with $M_- \neq X$ and $S|M$ nonzero. For such an M we can prove the following.

LEMMA 3.4. *Let $\mathscr{L}$ be a strongly reflexive subspace lattice, S a nonzero single element of* $\operatorname{Alg} \mathscr{L}$, *and $S|M$ nonzero for an $M \in \mathscr{L}$ with $M_- \neq X$. Then $S(X) \subseteq M$.*

Proof. Let $L \in \mathscr{L}$ satisfy $L \not\subseteq M$. We shall first show that $S(X) \subseteq L_-$.

If $L_- = X$ we have nothing to prove, so assume $L_- \neq X$. The condition $L \not\subseteq M$ implies $M \subseteq L_-$, so if $m \in M$ we have by the invariance of M under S

$$S(m) \in S(M) \subseteq M \subseteq L_-.$$

Let now $l^* \in L_-^\perp$ be arbitrary. Choose nonzero $l \in L$ and $m \in M$ with $S(m) \neq 0$ and $m^* \in M_-^\perp$ also nonzero. Then $l^* \otimes l$ and $m^* \otimes m$ belong to $\operatorname{Alg} \mathscr{L}$, and $S(m^* \otimes m) \neq 0$. However, $Sm \in L_-$ and $l^* \in L_-^\perp$, so

$$(l^* \otimes l) S(m^* \otimes m) = l^*(Sm) m^* \otimes l = 0.$$

The assumption that S is single implies $(l^* \otimes l)S = 0$. Hence for any $x \in X$

we have

$$l^*(Sx)l = (l^* \otimes l)Sx = 0$$

and so $Sx \in (L_-^{\perp})_{\perp} = L_-$. The arbitrariness of x in X implies $S(X) \subseteq L_-$, as required.

For the rest of the proof observe that the above shows that in fact

$$S(X) \subseteq \bigcap \{L_- \mid L \in \mathscr{L},\ L \not\subseteq M\}.$$

But by Lemma 2.2(i), the set on the right equals M. ■

Below we show how these last two lemmas can be used for alternative proofs of the results in [12] and [16] about single elements. To be specific, if $\mathscr{L}$ is a nest [16] or if it is a complete atomic Boolean subspace lattice [12], then every nonzero single element of $\operatorname{Alg}\mathscr{L}$ is of rank one. However, we shall have occasion to prove even stronger results. The first main theorem of this section is the following.

Theorem 3.5. *Let $\mathscr{L}$ be a strongly reflexive subspace lattice and S a single element of $\operatorname{Alg}\mathscr{L}$ such that $S^2 \neq 0$. Then S is of rank one. In particular, if $T \in \operatorname{Alg}\mathscr{L}$ is single, then the rank of T^2 is 0 or 1.*

Proof. As S^2 is nonzero, there is an $l_2 \in X$ such that $S^2 l_2 \neq 0$. Put $l_1 = Sl_2$ and $l = Sl_1$. We will show that $S(X) \subseteq \mathbb{C}l$. First we show that if $K \in \mathscr{L}$ is such that $K_- \neq X$, then $S(K) \subseteq \mathbb{C}l$. Indeed, if $S(K) = \{0\}$ we have nothing to prove. If instead $S(K) \neq \{0\}$, Lemma 3.4 implies $S(X) \subseteq K$. Hence $l_1 = S(l_2) \in K$. Note that $S|K$ is nonzero, since $Sl_1 = S^2 l_2 \neq 0$. But then Lemma 3.2 implies $S|K$ is of rank one. In particular $S(K) \subseteq \mathbb{C}Sl_1 = \mathbb{C}l$, as claimed. Denoting now by X_0 the linear span of $\{K \in \mathscr{L} \mid K_- \neq X\}$, the above shows that also $S(X_0) \subseteq \mathbb{C}l$. But X_0 is dense in X, so $S(X) = S(\overline{X_0}) \subseteq \overline{S(X_0)} \subseteq \overline{\mathbb{C}l} = \mathbb{C}l$, completing the proof. ■

It will be seen later that there exist single elements with $S^2 = 0$ and of rank greater than one. Indeed, the first example appears in [14], and we shall give others. Also notice that if S is a single element of a not necessarily unital algebra $\mathscr{A}$ containing it, the condition $S^2 \neq 0$ is equivalent to $S^3 \neq 0$ (if $S \cdot S \cdot S = 0$, then, for a single element, $S \cdot S$ is zero). So the condition $S^2 \neq 0$ implies $S\mathscr{A}S \neq \{0\}$. In the next theorem we show that any single element S of $\operatorname{Alg}\mathscr{L}$ with $S(\operatorname{Alg}\mathscr{L})S \neq \{0\}$, where $\mathscr{L}$ is a strongly reflexive subspace lattice, is of rank one. Moreover, we show by an example that $S(\operatorname{Alg}\mathscr{L})S \neq \{0\}$

is weaker than $S^2 \neq 0$, so the following variant of the previous theorem is a strengthening of it.

THEOREM 3.6. *Let $\mathscr{L}$ be a strongly reflexive subspace lattice, and $S \in \operatorname{Alg} \mathscr{L}$ a single element with $S(\operatorname{Alg} \mathscr{L})S \neq \{0\}$. Then S is of rank one.*

Proof. Let $A \in \operatorname{Alg} \mathscr{L}$ be such that $SAS \neq 0$, and let $l_2 \in X$ be such that $SASl_2 \neq 0$. Put $l_1 = ASl_2$ and $l = Sl_1$. For the rest of the proof we follow the notation in the previous theorem. As in Theorem 3.5, for $K \in \mathscr{L}$ with $K_- \neq X$ and $S|K$ nonzero we have $S(X) \subseteq K$, and so $Sl_2 \in K$. But then $l_1 = ASl_2 \in A(K) \subseteq K$, because $A \in \operatorname{Alg} \mathscr{L}$ and $K \in \mathscr{L}$. Hence, as $Sl_1 \neq 0$, it follows that $S(K) \subseteq \mathbb{C}Sl_1 = \mathbb{C}l$. From here on the proof runs identically to the one for Theorem 3.5. ■

We have shown that if S is single with $SAS \neq 0$ for some A in $\operatorname{Alg} \mathscr{L}$, then S is of rank one. Notice that the condition $SAS \neq 0$ for a single S implies $(AS)^2 \neq 0$. So writing $S = e^* \otimes f$, we have $0 \neq (AS)^2 = e^*(Af)AS$ and therefore $e^*(Af) \neq 0$. By scaling if necessary we may assume that $e^*(Af) = 1$ and so AS is a nonzero idempotent. But then $I - AS$ is not invertible [since $\operatorname{Ker}(I - AS) \supseteq \operatorname{range}(AS) \neq (0)$], and therefore $S \notin \operatorname{rad}(\operatorname{Alg} \mathscr{L})$. Conversely, if $S \in \operatorname{Alg} \mathscr{L}$ is a single with $S \notin \operatorname{rad}(\operatorname{Alg} \mathscr{L})$, then by a well-known property of the radical of a normed algebra, there is a $B \in \operatorname{Alg} \mathscr{L}$ with $I - BS$ not invertible. In particular $(BS)^2 \neq 0$ (otherwise $I + BS$ would be an inverse of $I - BS$), so $SBS \neq 0$, and Theorem 3.6 applies. Summarizing, we have shown the following:

THEOREM 3.7. *Let $\mathscr{L}$ be a strongly reflexive subspace lattice, and $S \in \operatorname{Alg} \mathscr{L}$ a single element not in the radical of $\operatorname{Alg} \mathscr{L}$. Then S is of rank one.*

It is tempting to state that if $\mathscr{L}$ is a strongly reflexive subspace lattice with $\operatorname{Alg} \mathscr{L}$ semisimple, then every nonzero single element of $\operatorname{Alg} \mathscr{L}$ is of rank one. This however has a much better formulation: It is proved in [10] that if $\mathscr{L}$ is a strongly reflexive subspace lattice, then $\operatorname{Alg} \mathscr{L}$ is semisimple if and only if $\mathscr{L}$ is a complete atomic Boolean subspace lattice. (The proof in [10] is given for Hilbert spaces but can be modified in the obvious way to hold for general normed spaces; see [12].) In particular we conclude that in complete atomic Boolean subspace lattices nonzero single elements are of rank one. There is a direct proof of this, which we now give.

COROLLARY 3.8 [12]. *If $\mathscr{L}$ is a complete atomic Boolean subspace lattice, then every single element of $\operatorname{Alg} \mathscr{L}$ is of rank one, and conversely.*

Proof. Let S be a single element of $\operatorname{Alg}\mathscr{L}$. By Lemma 3.2 there is an L with $L_- \neq X$ such that $S|L$ is of rank one, and by Lemma 3.4 $S(X) \subseteq L$. Since $L_- \neq X$, it follows that L is an atom of $\mathscr{L}$. If $M \in \mathscr{L}$ is any other atom (if $\mathscr{L}$ has no other atom, this step is unnecessary), we have $S(M) \subseteq S(X) \subseteq L$, and by invariance $S(M) \subseteq M$. Hence $S(M) \subseteq L \cap M = (0)$. Hence if X_0 is the (dense) linear span of atoms of $\mathscr{L}$, we have $S(X_0) = S(L)$, which is one dimensional. The result now follows from Lemma 2.4(i). ∎

COROLLARY 3.9 [16]. *If $\mathscr{L}$ is a nest, then every single element of* $\operatorname{Alg}\mathscr{L}$ *is of rank one, and conversely.*

Proof. Let S be a single element of $\operatorname{Alg}\mathscr{L}$. By Lemma 3.2 there exists an L in $\mathscr{L}$ such that $S(L)$ is one dimensional. Let $M \in \mathscr{L}$, $M_- \neq X$ be such that $M \supseteq L$. Since $S(M)$ is nonzero, it is also one dimensional. Clearly the one dimensional subspaces $S(M)$, $S(L)$ coincide, as $L \subseteq M$. Now, by the total order of $\mathscr{L}$, the set $X_0 = \bigcup\{M \in \mathscr{L} \mid M_- \neq X,\ M \supseteq L\}$ is a dense linear manifold, and by the above $S(X_0)$ is one dimensional. The result now follows from Lemma 2.4(i). ∎

REMARKS.

(1) Without the restrictions $S^2 \neq 0$, $S(\operatorname{Alg}\mathscr{L})S \neq \{0\}$ in Theorems 3.5 and 3.6, the conclusions may fail: In [14] there is an example of a strongly reflexive lattice $\mathscr{L}$, and a single element of $\operatorname{Alg}\mathscr{L}$ with rank equal to two. To be specific, $\mathscr{L}$ is $\{(0), \langle e_1\rangle, \langle e_3\rangle, \langle e_1, e_3\rangle, \langle e_1, e_2, e_3\rangle, \langle e_1, e_3, e_4\rangle, H\}$, where $H = \langle e_1, e_2, e_3, e_4\rangle$ for orthonormal e_1, e_2, e_3, e_4. It turns out that $\operatorname{Alg}\mathscr{L}$ consists of 4×4 matrices of the form

$$\begin{pmatrix} * & * & 0 & * \\ 0 & * & 0 & 0 \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}.$$

The single element in question is

$$S = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Direct computation shows that in this example we have $S(\operatorname{Alg}\mathscr{L})S = \{0\}$ (so also $S^2 = 0$). In the next section we will show the existence of single elements of infinite rank, and in these examples we have, once again, $S(\operatorname{Alg}\mathscr{L})S = \{0\}$.

(2) Just before Theorem 3.6 we claimed that, for single elements, the restriction $S(\operatorname{Alg}\mathscr{L})S \neq \{0\}$ is milder than $S^2 \neq 0$. For an example let $\mathscr{L} = \{(0), X\}$, in which case $\operatorname{Alg}\mathscr{L} = \mathscr{B}(X)$. Choose $e^* \in X^*$, $f \in X$, $A \in \mathscr{B}(X)$ such that $e^*(f) = 0$ and $e^*(Af) \neq 0$. Then $R = e^* \otimes f$ satisfies $R^2 = 0$, but $RAR \neq 0$. We remark that there exist strongly reflexive lattices that have $R^2 = 0$ for all rank one elements of $\operatorname{Alg}\mathscr{L}$. For example, any continuous nest ($L_- = L\ \forall L \in \mathscr{L}$) has this property. An example of a continuous nest is, in $L^2[0,1]$, the lattice consisting of all $L_a = \{f \in L^2[0,1] \mid f(x) = 0 \text{ for } x > a\}$ $(0 \leqslant a \leqslant 1)$.

(3) None of the theorems in Section 2 hold if the condition on the lattice is weakened to mere reflexivity rather than strong reflexivity. The obvious example is to choose $\mathscr{L}$ so that $\operatorname{Alg}\mathscr{L}$ has nonzero single elements but does not have any rank one operator. For example if $\mathscr{M}$ is a complete atomic Boolean subspace lattice (or a nest) and $\mathscr{L} = \operatorname{Lat}(\operatorname{Alg}\mathscr{M})^{(2)}$, then $\mathscr{L}$ is reflexive, $\operatorname{Alg}\mathscr{L}$ contains nonzero singles but contains no operators of rank one: $\operatorname{Alg}\mathscr{L} = \{A \oplus A \mid A \in \operatorname{Alg}\mathscr{M}\}$ (see [11]). However, we consider this example unfortunate, because the algebra $\operatorname{Alg}\mathscr{L} = (\operatorname{Alg}\mathscr{M})^{(2)}$ has a continuous faithful representation $\varphi : \operatorname{Alg}\mathscr{L} \to \operatorname{Alg}\mathscr{M}$, $A \oplus A \mapsto A$, in which the image of every nonzero single element is of rank one. We improve this example as follows:

Example 3.10. We show the existence of a reflexive lattice $\mathscr{L}$ such that $\operatorname{Alg}\mathscr{L}$ has single elements (and rank one operators) but with nonzero single elements of rank greater than one. Actually there exist nonzero single elements S, not of rank one, satisfying $S^2 \neq 0$, $S(\operatorname{Alg}\mathscr{L})S \neq \{0\}$. Finally we show that for suitable $\mathscr{L}$, the algebra $\operatorname{Alg}\mathscr{L}$ has *no* faithful representation in such a way that all single elements are carried to operators of rank one. More is proved: There is no faithful representation carrying single elements to compact operators. This is of interest in relation to [6]. An example is as follows:

Let X be an infinite dimensional normed space, and let $\mathscr{L}$ be the collection of subspaces of $X \oplus X$ given by $\mathscr{L} = \{L \oplus 0 \mid L \text{ a subspace of } X\} \cup \{X \oplus M \mid M \text{ a subspace of } X\}$. Here

$$\operatorname{Alg}\mathscr{L} = \begin{pmatrix} \lambda I & P \\ 0 & \mu I \end{pmatrix},$$

where λ, μ are scalars, I the identity on X, and $P \in \mathscr{B}(X)$ arbitrary.

Elementary calculations show that $\mathscr{L}$ is reflexive (see [13]) and that operators of the form

$$\begin{pmatrix} \lambda I & P \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & P \\ 0 & \mu I \end{pmatrix}$$

are single. Clearly then there exist single elements with $S^2 \neq 0$ and of rank greater than one. Suppose now that φ is a faithful representation that carries each single element of $\operatorname{Alg}\mathscr{L}$ to a compact operator on some normed space Y. Then as

$$\varphi\left(\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\right) + \varphi\left(\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}\right)$$

and each of the last two operators is compact as an image of a nonzero single, we conclude that the identity is mapped under φ to a compact operator K. For $A \in \operatorname{Alg}\mathscr{L}$ nonzero we have $\varphi(A) = \varphi(I)\varphi(A)\varphi(I) = K\varphi(A)K$. So the nonzero vector $\varphi(A)$ of $\mathscr{B}(Y)$ is an eigenvector of the (identity) transformation $T: \varphi(\operatorname{Alg}\mathscr{L}) \to \varphi(\operatorname{Alg}\mathscr{L})$, $\varphi(A) \mapsto K\varphi(A)K$. Using Theorem 3 of [2, p. 174], we conclude that T is compact, so $\varphi(A)$ belongs to the (finite dimensional) eigenspace of T belonging to the eigenvalue 1. In particular $\varphi(\operatorname{alg}\mathscr{L})$ is finite dimensional, which contradicts the fact that X, and hence $\operatorname{Alg}\mathscr{L}$, is infinite dimensional. The contradiction establishes the claim.

4. LATTICE CONDITIONS

In the previous section we studied algebraic conditions on a single element of $\operatorname{Alg}\mathscr{L}$ which imply that the element is of rank one. In spite of similarities, not all nonzero single elements in $\operatorname{Alg}\mathscr{L}$, for a strongly reflexive subspace lattice $\mathscr{L}$, are of rank one (Moore [14]). We have remarked that strongly reflexive subspace lattices in which every nonzero single element is of rank one include nests and complete atomic Boolean subspace lattices. However, Moore's example shows that even in CDC lattices (see [14] for the definition), nonzero single elements may fail to be of rank one.

In his example Moore considered, essentially, the "ordinal sum" $\mathscr{L}_1 + \mathscr{L}_2$ of two complete atomic Boolean lattices $\mathscr{L}_1$ and $\mathscr{L}_2$, and constructed a single element of $\operatorname{Alg}(\mathscr{L}_1 + \mathscr{L}_2)$ with rank two. In this section we produce a sharp upper bound for the rank of each single element of $\operatorname{Alg}(\mathscr{L}_1 + \mathscr{L}_2)$ in terms of the number of atoms of $\mathscr{L}_1$ and $\mathscr{L}_2$. As a result we show how to obtain single elements of any rank, including infinity. The motivation for the construction that appears in Theorem 5.3 is Moore's example.

Turning now to Question 2 of [14], which we partially answer, we show that if $\mathscr{L}_1, \mathscr{L}_2$ are strongly reflexive lattices in which every nonzero single element of Alg $\mathscr{L}_i$ is of rank one $(i = 1,2)$, then the same is true for the direct product $\mathscr{L}_1 \times \mathscr{L}_2$ and for the ordered product $\mathscr{L}_1 \geqslant \mathscr{L}_1$ (but not necessarily for $\mathscr{L}_1 \geqslant \mathscr{L}_2$). It will be apparent in this section that ordinal sums, direct products, and ordered products of lattices behave entirely differently as far as single elements are concerned, so that a full answer to Moore's question seems difficult.

Definition. Let $\mathscr{L}_1, \mathscr{L}_2$ be subspace lattices on a normed space X. Then the *ordinal sum* $\mathscr{L}_1 + \mathscr{L}_2$, the *ordered product* $\mathscr{L}_1 \geqslant \mathscr{L}_2$, and the *direct product* $\mathscr{L}_1 \times \mathscr{L}_2$ of $\mathscr{L}_1$ and $\mathscr{L}_2$ are defined as the sets of subspaces of $X \oplus X$ given by

$$\mathscr{L}_1 + \mathscr{L}_2 = \{L \oplus 0 \mid L \in \mathscr{L}_1\} \cup \{X \oplus M \mid M \in \mathscr{L}_2\},$$

$$\mathscr{L}_1 \geqslant \mathscr{L}_2 = \{L \oplus M \mid L \in \mathscr{L}_1,\ M \in \mathscr{L}_2, \text{ and } L \supseteq M\},$$

$$\mathscr{L}_1 \times \mathscr{L}_2 = \{L \oplus M \mid L \in \mathscr{L}_1,\ M \in \mathscr{L}_2\}.$$

Notice that all of these sets are subspace lattices. Also, $\mathscr{L}_1 + \mathscr{L}_2$ is a complete sublattice of $\mathscr{L}_1 \geqslant \mathscr{L}_2$, which in turn is a complete sublattice of $\mathscr{L}_1 \times \mathscr{L}_2$. In each of these lattices the lattice operations are "coordinatewise":

$$\bigvee_a (L_a \oplus M_a) = \Big(\bigvee_a L_a\Big) \oplus \Big(\bigvee_a M_a\Big),$$

$$\bigcap_a (L_a \oplus M_a) = \Big(\bigcap_a L_a\Big) \oplus \Big(\bigcap_a M_a\Big).$$

It is easy to see that if $\mathscr{L}_1$ and $\mathscr{L}_2$ are strongly reflexive, then so is $\mathscr{L}_1 \times \mathscr{L}_2$ and hence so are its complete sublattices $\mathscr{L}_1 \geqslant \mathscr{L}_2$ and $\mathscr{L}_1 + \mathscr{L}_2$.

Lemma 4.1. *If $\mathscr{L}_1, \mathscr{L}_2$ are subspace lattices, then the algebras of operators on $X \oplus X$ leaving invariant $\mathscr{L}_1 + \mathscr{L}_2$, $\mathscr{L}_1 \geqslant \mathscr{L}_2$, and $\mathscr{L}_1 \times \mathscr{L}_2$ are* *given by*

$$\mathrm{Alg}(\mathscr{L}_1 + \mathscr{L}_2) = \left\{ \begin{pmatrix} A & P \\ 0 & B \end{pmatrix} \middle| A \in \mathrm{Alg}\,\mathscr{L}_1,\ B \in \mathrm{Alg}\,\mathscr{L}_2,\ P \in \mathscr{B}(X) \right\},$$

$$\mathrm{Alg}(\mathscr{L}_1 \geqslant \mathscr{L}_2) = \left\{ \begin{pmatrix} A & P \\ 0 & B \end{pmatrix} \middle| A \in \mathrm{Alg}\,\mathscr{L}_1,\ B \in \mathrm{Alg}\,\mathscr{L}_2,\ P \in \mathscr{B} \right\},$$

where $\mathscr{B} \subseteq \mathscr{B}(X)$ *is given by* $\mathscr{B} = \{P \in \mathscr{B}(X) \mid P(M) \subseteq L$ *whenever* $L \in \mathscr{L}_1$, $M \in \mathscr{L}_2$, *and* $M \subseteq L\}$, *and*

$$\mathrm{Alg}(\mathscr{L}_1 \times \mathscr{L}_2) = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle| A \in \mathrm{Alg}\,\mathscr{L}_1,\ B \in \mathrm{Alg}\,\mathscr{L}_2 \right\}.$$

Proof. The proof is routine, and we only show it for the algebra $\mathrm{Alg}(\mathscr{L}_1 \geqslant \mathscr{L}_2)$: If $\begin{pmatrix} A & P \\ C & B \end{pmatrix}$ belongs to $\mathrm{Alg}(\mathscr{L}_1 \geqslant \mathscr{L}_2)$ then the invariance of $X \oplus 0$ shows $C = 0$. The invariance of elements of the form $L \oplus 0$ $(L \in \mathscr{L}_1)$ or of the form $X \oplus M$ $(M \in \mathscr{L}_2)$ show that $A \in \mathrm{Alg}\,\mathscr{L}_1$ and $B \in \mathrm{Alg}\,\mathscr{L}_2$. If now $L \oplus M \in \mathscr{L}_1 \geqslant \mathscr{L}_2$ and $m \in M$, then $0 \oplus m \in L \oplus M$, so

$$\begin{pmatrix} Pm \\ Bm \end{pmatrix} = \begin{pmatrix} A & P \\ 0 & B \end{pmatrix} \begin{pmatrix} 0 \\ m \end{pmatrix} \in \begin{pmatrix} A & P \\ 0 & B \end{pmatrix} (L \oplus M) \subseteq L \oplus M,$$

that is, $Pm \in L$. The arbitrariness of m in M shows $P(M) \subseteq L$. The rest is even easier. ■

Notice that, as expected, $\mathrm{Alg}(\mathscr{L}_1 + \mathscr{L}_2) \supseteq \mathrm{Alg}(\mathscr{L}_1 \geqslant \mathscr{L}_2) \supseteq \mathrm{Alg}(\mathscr{L}_1 \times \mathscr{L}_2)$.

Before stating the next two theorems, we remark that both are concerned with the ordinal sum of two strongly reflexive lattices. However, for the subspace lattices considered in Theorem 4.3, we show the existence of single elements of any rank. By contrast, in Theorem 4.2, we show that if one of the summands is a nest, then nonzero single elements can only be of rank one.

THEOREM 4.2. *If* $\mathscr{L}_1, \mathscr{L}_2$ *are subspace lattices with one of them a nest and the other any strongly reflexive subspace lattice, then every nonzero single element of* $\mathrm{Alg}(\mathscr{L}_1 + \mathscr{L}_2)$ *is of rank one.*

Proof. First let us assume that $\mathscr{L}_2$ is a nest. Let $X_0 = \text{linear span}\{K \in \mathscr{L}_1 + \mathscr{L}_2 \mid K_- \neq X \oplus X\}$, which is dense. It is easy to see that if $L \in \mathscr{L}_1$,

$M \in \mathscr{L}_2$, then in $\mathscr{L}_1 + \mathscr{L}_2$ we have $(L\oplus 0)_- = L_-\oplus 0$, $(X\oplus M)_- = X\oplus M_-$, so that $X_0 = \text{linear span}\{X\oplus M \mid M \in \mathscr{L}_2,\ M_- \neq X\}$. The total order of $\mathscr{L}_2$ shows that in fact $X_0 = \bigcup\{X\oplus M \mid M \in \mathscr{L}_2,\ M_- \neq X\}$.

Let now $x, y \in X_0$. We show that Sx, Sy are linearly dependent. Indeed, the conditions $x, y \in X_0$ imply the existence of an $M \in \mathscr{L}_2$, $M_- \neq X$ such that $x, y \in X\oplus M$. By Lemma 3.2, the operator $S|X\oplus M$ is of rank one or zero, so Sx and Sy are linearly dependent. This shows that $S(X_0)$ is one dimensional and hence, by Lemma 2.4(i), so is $S(X\oplus X)$, as required.

If on the other hand $\mathscr{L}_1$ is a nest, then a similar argument, but now using Lemma 3.3, shows that $S^*|X_1$, where $X_1 = \text{linear span}\{K_-^\perp \mid K \in \mathscr{L}_1 + \mathscr{L}_2,\ K \neq 0,\ K_- \neq X\oplus X\}$, is of rank one. By Lemma 2.4(ii) and the weak* density of X_1 it follows that S^*, and hence S, is of rank one. ■

We now show that if none of the summands is a nest, the situation can be drastically different. In what follows we allow n_1, n_2 to be infinite. Also we state the theorem for Banach spaces to avoid convergence technicalities, although this is not essential.

THEOREM 4.3. *Let $\mathscr{L}_1, \mathscr{L}_2$ be complete atomic Boolean subspace lattices on a Banach space with numbers of atoms n_1 and n_2 respectively. Then every single element of Alg$(\mathscr{L}_1 + \mathscr{L}_2)$ is of rank less than or equal to* $\min(n_1, n_2)$, *and this estimate is sharp.*

Proof. If n_1, n_2 are both infinite, then one of the claims in the theorem is automatically satisfied, so all that remains to be done, in this case, is to produce a single element of infinite rank.

Let L_i $(i \in \mathbb{N})$, M_i $(i \in \mathbb{N})$ be two infinite sequences of distinct atoms from $\mathscr{L}_1$ and $\mathscr{L}_2$ respectively. Choose nonzero vectors $e_i \in M_i$, $e_i^* \in (M_i')^\perp$ such that $e_i^*(e_i) \neq 0$; also choose $u_i \in L_i$ nonzero.

Let (a_{ij}) be an infinite matrix with the following properties:

(i) $a_{ij} \neq 0$ for all $i, j \in \mathbb{N}$,

(ii) the rows of (a_{ij}) are linearly independent (with coordinatewise operations),

(iii) $y_i = \Sigma a_{ij} u_j$ converges for each $i \in \mathbb{N}$,

(iv) $T = \Sigma e_i^* \otimes y_i$ converges.

There is an abundance of matrices satisfying (i) and (ii). By suitably scaling u_i and e_i^* if necessary, properties (iii) and (iv) can also be guaranteed. [In l_2, for example, (a_{ij}) could be chosen to be the Hilbert matrix $a_{ij} = 1/(i+j)$ and u_i assumed to be of norm i^{-1}. In this case the sum in

(iii) converges because $\|y_i\| \leqslant \Sigma |a_{ij}| j^{-1} \leqslant \Sigma j^{-2}$ and so the sum in (iv) converges if $\|e_i^*\| = 1/2^i$.]

We shall show that the operator $\begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}$ of $\mathrm{Alg}(\mathscr{L}_1 + \mathscr{L}_2)$ is single and of infinite rank.

Applying T to e_i, we get $Te_i = y_i$. By (ii) the y_i are linearly independent, so the range of T, and hence of $\begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}$, is infinite dimensional.

To prove that $\begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}$ is single, suppose

$$\begin{pmatrix} A & P \\ 0 & B \end{pmatrix}\begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}\begin{pmatrix} C & Q \\ 0 & D \end{pmatrix} = 0,$$

or equivalently $ATD = 0$. We shall show that either AT or TD is zero and hence that either

$$\begin{pmatrix} A & P \\ 0 & B \end{pmatrix}\begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}\begin{pmatrix} C & Q \\ 0 & D \end{pmatrix}$$

is zero. Recall that $A \in \mathrm{Alg}\,\mathscr{L}_1$ and $D \in \mathrm{Alg}\,\mathscr{L}_2$.

If $D^*e_i^*$ is zero for each $i \in \mathbb{N}$, then

$$TD = \sum D^*e_i^* \otimes y_i = 0.$$

If instead there is an i with $D^*e_i^* \neq 0$, then by the density of the linear span of the set of atoms of $\mathscr{L}_2$ there is an atom $M \in \mathscr{L}_2$ and an $x \in M$ such that $e_i^*(Dx) = D^*e_i^*(x) \neq 0$. But D leaves M invariant, so $Dx \in M$. Now for any atom $L \in \mathscr{L}_2$ other than M_i we have $L \subseteq M_i'$, so $M_i'^{\perp} \subseteq L^{\perp}$. As $e_i^* \in M_i'^{\perp}$, it follows that $L \subseteq \mathrm{Ker}(e_i^*)$, and so, as $M \not\subseteq \mathrm{Ker}(e_i^*)$, we have $M = M_i$. Therefore x and Dx belong to M_i. Hence for $j \neq i$ we have $e_j^*(Dx) \in e_j^*(M_i) = \{0\}$.

Now $\mathrm{Ker}(e_i^*)$ is of codimension 1 in X, and $e_i \notin \mathrm{ker}(e_i^*)$, so for some scalar λ we have

$$Dx = \lambda e_i + n,$$

where $n \in \mathrm{ker}(e_i^*)$. Note that $\lambda \neq 0$, because $e_i^*(Dx) \neq 0$. Applying x to the relation $ATD = 0$, we obtain

$$\begin{aligned} 0 = ATDx &= A\sum e_j^*(Dx)y_j \\ &= A[e_i^*(Dx)y_i] \\ &= e_i^*(\lambda e_i + n)Ay_i \\ &= \sum a_{ij}Au_j, \end{aligned}$$

from which it follows that $\Sigma a_{ij}Au_j = 0$.

Notice that as $A \in \operatorname{Alg}\mathscr{L}_1$ and $u_j \in L_j \in \mathscr{L}_1$, we have $Au_j \in L_j$. But also

$$a_{ij}Au_j = -\sum_{k \neq j} a_{ik}Au_k \in \bigvee_{k \neq j} L_k = L_j'.$$

Combining this with $Au_j \in L_j$, we find $a_{ij}Au_j \in L_j \cap L_j' = (0)$. Now the a_{ij} were assumed nonzero, so $Au_j = 0$ ($j \in \mathbb{N}$), and so $Ay_i = 0$ ($i \in N$). This then shows that $AT = e_i^* \otimes Ay_i = 0$, as required.

The above exhibits a single element of infinite rank if both $\mathscr{L}_1$ and $\mathscr{L}_2$ have an infinite numbers of atoms. If one of $\mathscr{L}_1, \mathscr{L}_2$ has a finite number of atoms, then putting $k = \min(n_1, n_2)$, we have $k \in \mathbb{N}$. The above construction now runs the same, but instead of an infinite matrix we take a $k \times k$ matrix (a_{ij}) $(l < i, j < k)$ and make the obvious modifications.

Summing up, we have produced a single element of $\operatorname{Alg}(\mathscr{L}_1 + \mathscr{L}_2)$ of rank $\min(n_1, n_2)$. We now show that every single element has rank less than or equal to $\min(n_1, n_2)$. Having done that, the proof will be complete.

Let $S \in \operatorname{Alg}(\mathscr{L}_1 + \mathscr{L}_2)$ be single. We first show that the rank of S is less than or equal to n_2.

As $\mathscr{L}_2$ has n_2 atoms, there exist at most n_2 subspaces of the form $X \oplus L$ (L an atom of $\mathscr{L}_2$) with $S(X \oplus L)$ nonzero. An easy verification shows that $(X \oplus L)_- = X \oplus L_-$, so, for an atom L, we have $(X \oplus L)_- \neq X \oplus X$. By Lemma 3.2 $S|(X \oplus L)$ is of rank one. Denoting the linear manifold linear span$\{X \oplus L \mid L$ an atom of $\mathscr{L}_2\}$ by X_0, we have

$$\begin{aligned} S(X_0) &= \text{linear span}\{S(X \oplus L) \mid L \text{ an atom of } \mathscr{L}_2\} \\ &= \text{linear span}\{S(X \oplus L) \mid L \text{ an atom of } \mathscr{L}_2 \text{ and } S(X \oplus L) \neq 0\}; \end{aligned}$$

hence $\dim S(X_0) \leqslant \Sigma \dim S(X \oplus L) \leqslant n_2$.

By Lemma 2.4(i), $S(X)$ also has dimension less than or equal to n_2, as required. It remains to show that the rank of S is less than or equal to n_1, or equivalently, that the rank of S^* is less than or equal to n_1.

Let L be an atom of $\mathscr{L}_1$. We have $(L \oplus 0)_- = L_- \oplus 0$, so $(L \oplus 0)_-^\perp = L_-^\perp \oplus X^*$. By Lemma 3.3, $S^*|(L_-^\perp \oplus X^*)$ is either zero or of rank one, and the proof that the dimension of $S^*(X^* \oplus X^*)$ is less than or equal to n_1 runs as above, but with the (weak* dense) linear manifold $X_1 = \text{linear span}\{L_-^\perp \oplus X^* \mid L \text{ an atom of } \mathscr{L}_1\}$ instead of X_0 and using Lemma 2.4(ii) instead of Lemma 2.4(i). ∎

Let us now turn to the direct product of two subspace lattices. Recall that subspace lattices which have the property that every nonzero single element is of rank one include nests, complete atomic Boolean lattices, and the ones given in Theorem 4.1. Here is an easy way to produce a new class of such lattices, which are completely unlike the previous ones.

THEOREM 4.4. *Let $\mathscr{L}_1, \mathscr{L}_2$ be (not necessarily strongly reflexive) subspace lattices. Then every nonzero single element of* $\operatorname{Alg}(\mathscr{L}_1 \times \mathscr{L}_2)$ *is of rank one if, and only if, the same is true for the nonzero single elements of both* $\operatorname{Alg}\mathscr{L}_1$ *and* $\operatorname{Alg}\mathscr{L}_2$.

Proof. If S [respectively T] is a nonzero single element of $\operatorname{Alg}\mathscr{L}_1$ [respectively $\operatorname{Alg}\mathscr{L}_2$], then so is

$$\begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \qquad \left[\text{respectively} \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}\right]$$

for $\operatorname{Alg}(\mathscr{L}_1 \times \mathscr{L}_2)$, so one direction of the theorem is trivial. The converse is also elementary. If $\begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}$ is a nonzero single element of $\operatorname{Alg}(\mathscr{L}_1 \times \mathscr{L}_2)$, then if I denotes the identity on X, the equality

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} = 0$$

and the fact that $\begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}$ is single show that either S or T is zero. Hence single elements of $\operatorname{Alg}(\mathscr{L}_1 \times \mathscr{L}_2)$ are of the form $\begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}$. Considering products of the form

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}\begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix},$$

we conclude that both S and T are single elements of their respective algebras and hence of rank one, completing the proof. ■

We have showed that the above property of single elements is inherited from two lattices $\mathscr{L}_1, \mathscr{L}_2$ to their direct product and conversely. For ordinal sums neither of these inheritances holds. Indeed, Theorem 4.3 shows that $\mathscr{L} + \mathscr{L}$ may fail to have the property although $\mathscr{L}$ itself has it, and Theorem 4.2 shows that $\mathscr{L}$ may fail to have the property but $\mathscr{N} + \mathscr{L}$, for a nest $\mathscr{N}$, never fails. In the case of ordered products the situation is different. If $\mathscr{L}_1$ is not equal to $\mathscr{L}_2$, examples show that even if every single element of $\operatorname{Alg}(\mathscr{L}_1 \geqslant \mathscr{L}_2)$ is of rank one, it does not follow that the same is true for the single elements of $\operatorname{Alg}\mathscr{L}_1$ or $\operatorname{Alg}\mathscr{L}_2$. The converse also fails, even if one of the lattices is a nest. It seems that one of the difficulties when considering $\operatorname{Alg}(\mathscr{L}_1 \geqslant \mathscr{L}_2)$ with $\mathscr{L}_1$ not equal to $\mathscr{L}_2$ is that this algebra consists of operators of the form

$$\begin{pmatrix} A & P \\ 0 & B \end{pmatrix}$$

where, by Lemma 4.1, P belongs to some set $\mathscr{B}$ which is not in general an algebra (see Example 4.7). On the other hand, if $\mathscr{L}_1 = \mathscr{L}_2$, say equal to $\mathscr{L}$, it is easy to see that $\mathscr{B} = \operatorname{Alg}\mathscr{L}$ and so

$$\operatorname{Alg}(\mathscr{L} \geqslant \mathscr{L}) = \left\{ \begin{pmatrix} A & P \\ 0 & B \end{pmatrix} \middle| A, P, B \in \operatorname{Alg}\mathscr{L} \right\}.$$

In this case it is easy to see that if single elements of $\operatorname{Alg}(\mathscr{L} \geqslant \mathscr{L})$ are of rank one, then the same is true for $\operatorname{Alg}\mathscr{L}$. We show that the converse is true for strongly reflexive subspace lattices.

THEOREM 4.5. *Let $\mathscr{L}$ be a strongly reflexive subspace lattice in which every single element is of rank one. Then every single element of* $\operatorname{Alg}(\mathscr{L} \geqslant \mathscr{L})$ *is of rank one.*

Proof. Let

$$\begin{pmatrix} S & T \\ 0 & U \end{pmatrix}$$

be a single element of $\operatorname{Alg}(\mathscr{L} \geqslant \mathscr{L})$. Routine calculations, which we omit, show that S, T, U will all be single elements of Alg $\mathscr{L}$. Also, starting from

$$\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} S & T \\ 0 & U \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = 0,$$

we conclude that either S or U is zero. Hence the nonzero single elements of Alg$(\mathscr{L} \geqslant \mathscr{L})$ are of the form

$$\begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 0 & U \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} S & T \\ 0 & 0 \end{pmatrix}$$

or finally

$$\begin{pmatrix} 0 & T \\ 0 & U \end{pmatrix}.$$

(Before continuing, we mention that in the last two cases *it is* possible to have single elements with S, T, U nonzero.)

By assumption each of the single elements S, T, U of Alg $\mathscr{L}$ is of rank one, say $S = s^* \otimes s$, $T = t^* \otimes t$, and $U = u^* \otimes u$, so the first three types of single elements listed above are of rank one. We proceed to show that the same is true for the other two types; let us start with $\begin{pmatrix} 0 & T \\ 0 & U \end{pmatrix}$.

We now claim that $T + U$ is of rank one. If not, then $T + U$ would not be single and there would exist A and B with $A(T+U)B = 0$ and $A(T+U) \neq 0 \neq (T+U)B$. But then we would have

$$\begin{pmatrix} A & A \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & T \\ 0 & U \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} = 0$$

with neither

$$\begin{pmatrix} A & A \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & T \\ 0 & U \end{pmatrix} \quad \text{nor} \quad \begin{pmatrix} 0 & T \\ 0 & U \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$$

equal to zero. This contradiction establishes the claim. We further show that u^* and t^* must be dependent. Indeed, if not, then as $T + U = t^* \otimes t + u^* \otimes u$ is of rank one, we would have that t and u are linearly dependent, say $t = \lambda u$ for some nonzero λ. Pick now, by the density of X_0 after Lemma 2.3, an L with $L_- \neq X$ and a $q \in L$ with $t^*(q) = 1$, and let $p^* \in L_-^\perp$. Thus $p^* \otimes q \in \operatorname{Alg} \mathscr{L}$ and

$$\begin{pmatrix} u^*(q)I & -\lambda I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & T \\ 0 & U \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & p^* \otimes q \end{pmatrix} = 0.$$

But clearly

$$\begin{pmatrix} 0 & T \\ 0 & U \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & p^* \otimes q \end{pmatrix} \neq 0.$$

Also

$$\begin{pmatrix} u^*(q)I & -\lambda I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & T \\ 0 & U \end{pmatrix} = \begin{pmatrix} 0 & u^*(q)T - \lambda U \\ 0 & 0 \end{pmatrix}$$

is nonzero by the linear independence of t^* and u^*. This contradiction to the singleness of $\begin{pmatrix} 0 & T \\ 0 & U \end{pmatrix}$ shows that after all u^* and t^* must be linearly dependent, say $u^* = \mu t^*$. Hence for any $x, y \in X$ we have

$$\begin{pmatrix} 0 & T \\ 0 & U \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t^*(y)t \\ \mu t^*(y)u \end{pmatrix} = t^*(y) \begin{pmatrix} t \\ \mu u \end{pmatrix},$$

showing that the range of $\begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}$ is the one dimensional subspace spanned by $t \oplus \mu u$, showing in turn that this operator is of rank one.

In the case of single elements of the form

$$\begin{pmatrix} S & T \\ 0 & 0 \end{pmatrix},$$

a similar argument starting from

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S & T \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & B \\ 0 & B \end{pmatrix}$$

shows that $S+T$ must be of rank one. Again a similar argument shows that s and t must be linearly dependent, say $s=\mu t$ for some scalar μ. We now have for any x, y in X

$$\begin{pmatrix} S & T \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s^*(x)s+\mu t^*(y)s \\ 0 \end{pmatrix} = [s^*(x)+\mu t^*(y)]\begin{pmatrix} s \\ 0 \end{pmatrix},$$

showing that the range of the single element is one dimensional, and concluding the proof. ∎

EXAMPLE 4.6. We show that if $\mathscr{L}_1, \mathscr{L}_2$ are different lattices, it may happen that $\operatorname{Alg}(\mathscr{L}_1 \geqslant \mathscr{L}_2)$ has single elements of rank greater than one. In fact this can happen even for two (lattice isomorphic) complete atomic Boolean subspace lattices of the simplest kind. The converse may also fail in that it may happen that all nonzero single elements of $\operatorname{Alg}(\mathscr{L}_1 \geqslant \mathscr{L}_2)$ are of rank one, but not so for those of $\operatorname{Alg}\mathscr{L}_1$ and $\operatorname{Alg}\mathscr{L}_2$.

For the one direction take $\mathscr{L}_1, \mathscr{L}_2$ to be complete atomic Boolean subspace lattices $\{(0), K, L, X\}$ and $\{(0), M, N, X\}$ respectively, where the K, L, M, N are incomparable in pairs (e.g., in $\mathbb{C}^2$, we could take four different lines through 0). In this case $\mathscr{L}_1 \geqslant \mathscr{L}_2 = \{0\oplus 0, K\oplus 0, L\oplus 0, X\oplus 0, X\oplus M, X\oplus N, X\oplus X\}$, and so $\mathscr{L}_1 \geqslant \mathscr{L}_2$ is the same as $\mathscr{L}_1 + \mathscr{L}_2$. By Theorem 4.3 there exist single elements in $\operatorname{Alg}(\mathscr{L}_1 \geqslant \mathscr{L}_2)$ of rank 2, a property not shared by $\operatorname{Alg}\mathscr{L}_1$ or $\operatorname{Alg}\mathscr{L}_2$. Finally, observe that in the above example, $\mathscr{L}_1$ and $\mathscr{L}_2$ are different but are isomorphic: that is, different as subspace lattices but the same as lattices.

Let us now give an example to show that the other direction of the theorem may fail if we do not assume $\mathscr{L}_1 = \mathscr{L}_2$: Take $\mathscr{L}_1 = \{(0), X\}$ and $\mathscr{L}_2$ a subspace lattice to be chosen presently. Here it is easy to see that $\mathscr{L}_1 \geqslant \mathscr{L}_2 = \mathscr{L}_1 + \mathscr{L}_2$, and, as $\mathscr{L}_1$ is a nest, Theorem 4.2 shows that every nonzero single in $\operatorname{Alg}(\mathscr{L}_1 \geqslant \mathscr{L}_2)$ is of rank one. For suitable $\mathscr{L}_2$, for instance one given by Theorem 4.3, we can arrange that the stated property of singles does not pass down to $\mathscr{L}_2$.

EXAMPLE 4.7. Just before Theorem 4.5 we stated that the set

$$\mathscr{B} = \{P \in (X) \mid P(M) \subseteq L \text{ whenever } L \in \mathscr{L}_1,\ M \in \mathscr{L}_2, \text{ and } M \subseteq L\},$$

which appears in Lemma 4.1, is not in general an algebra. For an example let

e_1, e_2, e_3 be the three standard basis vectors of $\mathbb{C}^3$; put $\mathscr{L}_1 = \{(0), \langle e_1, e_2\rangle, \mathbb{C}^3\}$, and $\mathscr{L}_2 = \{(0), \langle e_1\rangle, \mathbb{C}^3\}$. Therefore $\mathscr{B} = \{P \in \mathscr{B}(X) \mid Pe_1 \in \langle e_1, e_2\rangle\}$, and hence the transformation $T(x,y,z) = (0,x,y)$ belongs to $\mathscr{B}$. However, $T^2 \notin \mathscr{B}$, as $T^2(1,0,0) = (0,0,1) \notin \langle e_1, e_2\rangle$.

We remark that the set $\mathscr{B}$ always satisfies $\operatorname{Alg} \mathscr{L}_i \subseteq \mathscr{B} \subseteq \operatorname{Alg}(\mathscr{L}_1 \cap \mathscr{L}_2)$ $(i = 1,2)$. Indeed, if $A \in \operatorname{Alg} \mathscr{L}_1$ and $L \in \mathscr{L}_1$, $M \in \mathscr{L}_2$, $M \subseteq L$, we have $A(M) \subseteq A(L) \subseteq L$. If instead $A \in \operatorname{Alg} \mathscr{L}_2$, we have $A(M) \subseteq M \subseteq L$. Combining the two, we get $\operatorname{Alg} \mathscr{L}_i \subseteq \mathscr{B}$. Also, if $P \in \mathscr{B}$ and $K \in \mathscr{L}_1 \cap \mathscr{L}_2$, then as $K \in \mathscr{L}_1$, $K \in \mathscr{L}_2$, and $K \subseteq K$, we have $P(K) \subseteq K$, showing that $P \in \operatorname{Alg}(\mathscr{L}_1 \cap \mathscr{L}_2)$. In particular it follows that if $\mathscr{L}_1 \supseteq \mathscr{L}_2$ or if $\mathscr{L}_2 \supseteq \mathscr{L}_1$, then $\mathscr{B} = \operatorname{Alg} \mathscr{L}$, where $\mathscr{L}$ is the smaller of $\mathscr{L}_1, \mathscr{L}_2$. We used a special case of this when $\mathscr{L}_1 = \mathscr{L}_2$ in Theorem 4.5.

We wish to thank Dr. J. A. Erdos for discussions relating to this work, particularly for suggesting the problem and for permission to include Lemma 3.2. Also we wish to thank Professor S. Papadopoulou for shortening the original proof of Theorem 3.5.

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Received 18 July 1989; final manuscript accepted 18 December 1989